The Art Gallery Problem

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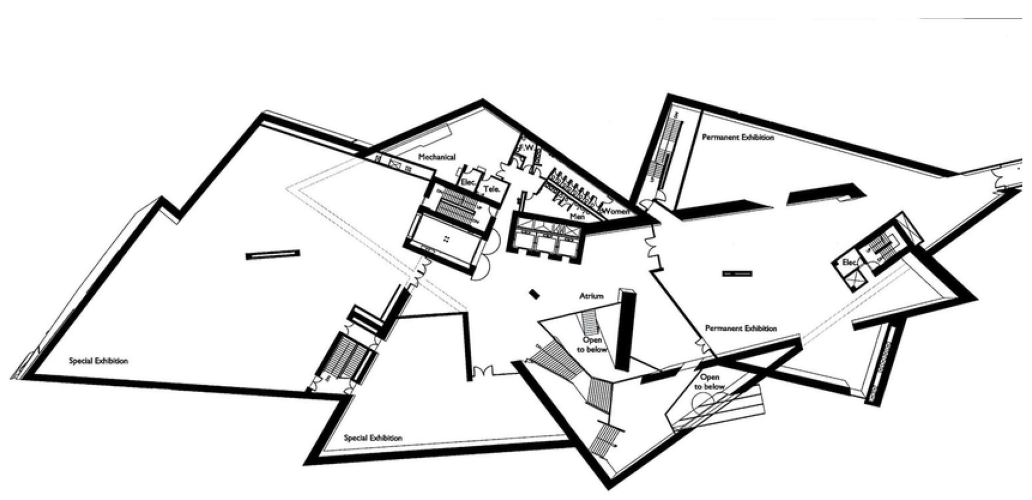
1. Introduction

What is the maximum number of guards required to monitor a polygonal art gallery? This geometric problem, known as The Art Gallery Problem, was introduced by Victor Klee in 1973. Through viewing the floor of an art gallery as a closed polygon, Geometry and Graph Theory can be used to determine the number of guards necessary. The guards are stationary figures and have a complete field of view but not through walls or around corners. For efficiency, the minimum number of guards a gallery requires to view the entire floor is desired. The Art Gallery Problem provides the maximum number of guards essential for a complete monitoring of any gallery with n walls.



2. Triangulation and 3-coloring Comes to Aid

Understanding our problem: We want to find the maximum number of guards required to efficiently monitor an n-sided polygonal gallery.

Suppose we have an n-sided polygon we need to guard. Consider the 10-sided polygon shown in Figure 1.

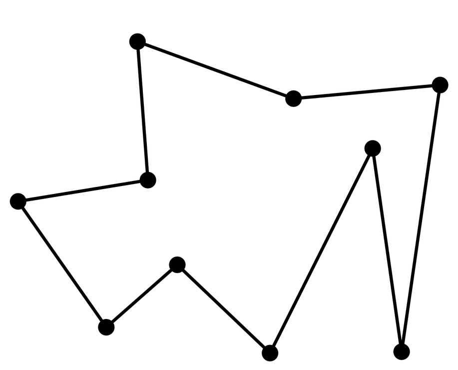


Figure 1

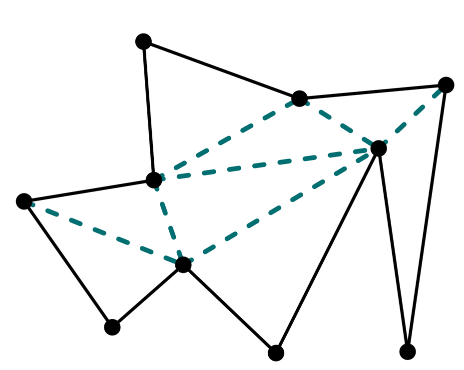


Figure 2

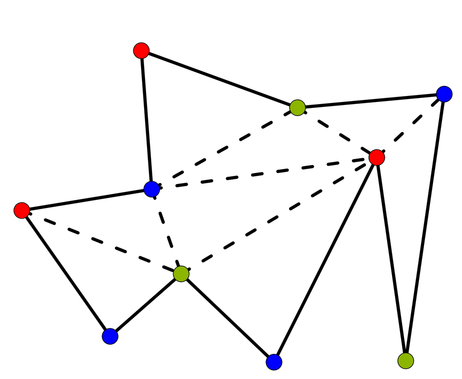


Figure 3

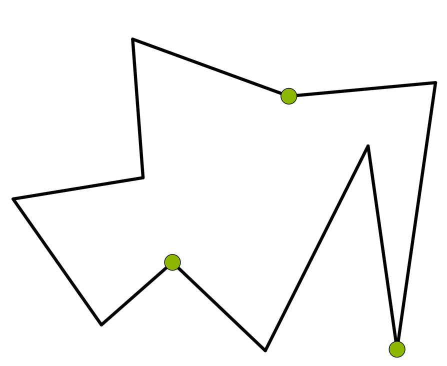


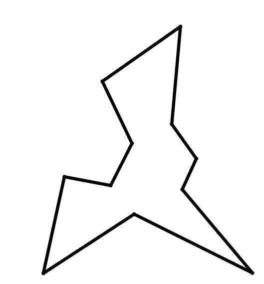
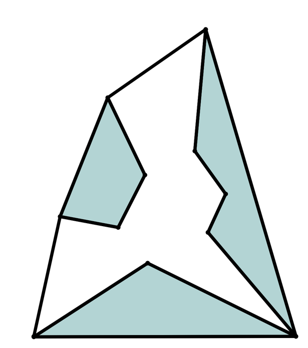
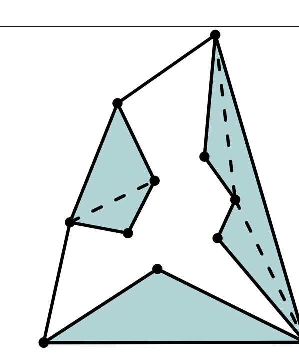
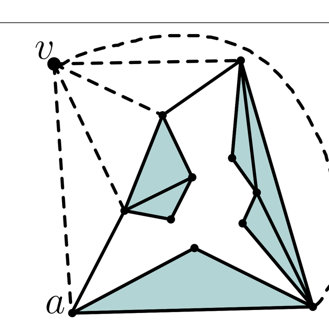
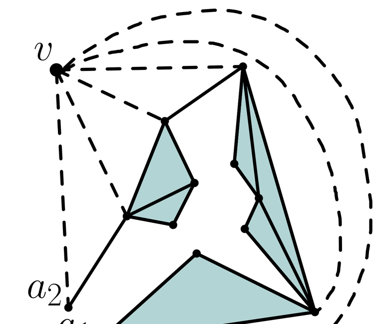
Figure 4

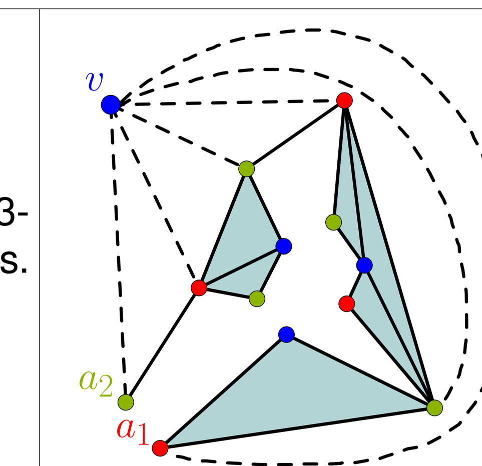
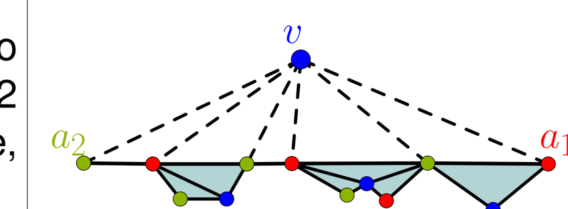
1. The first step required is to triangulate the polygon as shown in Figure 2.
2. The next step is to color each vertex in a way that no adjacent vertices share the same color (Figure 3).
3. Placing a guard at each green or red point will guarantee the full protection of the gallery as shown in Figure 4.

Theorem: For art galleries with n walls, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are sufficient and sometimes necessary.

3. Guarding Fortresses

To prevent an outside attack on a fortress, guards must be positioned along the exterior. Every point on the perimeter must be visible by at least one guard.

Consider this 11-sided polygon.	
We begin by forming a convex hull. This is the smallest convex polygon outside the fortress but inside the convex hull. The shaded areas are the hull pockets. This area is outside the fortress and inside the convex hull.	
The next step is to triangulate each hull pocket.	
Then place a new vertex named v away from the fortress. Connect each vertex of the convex hull to vertex v.	
Split the vertex a into 2 vertices named a_1 and a_2 . The unbounded region is the interior of the fortress while the bounded region is the hull pockets.	

Then we need to find the proper 3-coloring of the bounded region vertices. .	
The bounded region can be unfolded to form a triangulation of a polygon with n+2 vertices. Now we have 5 green, 4 blue, and 4 red vertices.	

The line a_1 to a_2 is the exterior of the convex hull. The guards at the blue vertices completely monitor the exterior. Since the hull pockets are triangulated, the blue vertices also monitor the hull pockets. Similarly, guards at the red vertices also protect the fortress.

The vertices along the line a_1 to a_2 alternate red and green, and there are at most n+1 vertices. Therefore one of the colors occurs at most

$$\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \text{ times.}$$

Theorem: For fortresses with n walls, $\left\lceil \frac{n}{2} \right\rceil$ guards are sufficient and sometimes necessary.

4. Conclusion

The Art Gallery Problem provides a solution to find the maximum number of guards needed to protect the interior of an art gallery. We have the resulting maximum of $\frac{n}{3}$ guards for any n-sided polygon. A reduction in guards can made when one focuses on certain subsets of polygons or guards. We have discussed the general problem and the extension of guarding the perimeter. Further extensions include introducing cameras with 180° vision. The number of guards required with 180° vision has yet to be proven.

There are various applications of this idea. One application is the Light Up Akari puzzle where a player has to place lights in order to illuminate an entire polygon with given obstacles. Our problem is applicable to many ideas and can be further extended through studying specific types of polygons and areas with increased intricacy.

References

[1] Michael, T. S. “Guards, Galleries, Fortresses, and the Octoplex.” The College Mathematics Journal, vol. 42, no. 3, 2011, pp. 191–199., https://doi.org/10.4169/college.math.j.42.3.191.