From Lagrange's Identity to the House of Representatives

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1. Motivation

Joseph-Louis Lagrange (1736-1813) is considered one of the greatest mathematicians of all times. His well-known contributions include Lagrange multiplier and Lagrange's identity, a tool to optimize functions subject to constraints. Many important results were derived directly from Lagrange's identity. We will focus on a real-world application, the method of equal proportions for apportionment of the United States House of Representatives.

The United States House of Representatives has the total number of 435 seats which is apportioned between 50 states. The U.S. Constitution guarantees a minimum of one representative for each state, while the Congress decides the calculation method to allocate the remaining 385 seats. The desire is to find a method that would provide the most accurate allocation based on each state's share of the national population. The problem arises as the state's share is not an integer.

Let's define the following notation:

- P_i = number of people in state i,
- H_i = number of seats to be assigned to state i,
- $P = \sum_{i=1}^{50} P_i$ = the total population of the states,
- $H = \sum_{i=1}^{50} H_i$ = the fixed total number of seats in the House.

The method currently used is called The Method of Equal Proportions.

2. Lagrange's Identity

For any two ordered pairs of real numbers (a_1, b_1) and (a_2, b_2),

$$({a_1}^2 + {a_2}^2)({b_1}^2 + {b_2}^2) - (a_1b_1 + a_2b_2)^2 = (a_1b_2 - a_2b_1)^2.$$

Similarly, for any three ordered pairs of real numbers (a_1, b_1), (a_2, b_2), and (a_3, b_3), we get

$$\begin{aligned}({a_1}^2 + {a_2}^2 + {a_3}^2)({b_1}^2 + {b_2}^2 + {b_3}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ = (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2.\end{aligned}$$

By generalizing the equations above for any n ordered pairs of real numbers (a_1, b_1),...,(a_n, b_n), the Lagrange's identity is

$$\left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right) - \left(\sum_{i=1}^{n} {a_i}^2{b_i}^2\right) = \sum_{1\le i<j\le n} (a_ib_i - a_jb_i)^2,$$

which can be proven by mathematical induction.

3. The Method of Equal Proportions Algorithm

The goal of this method is to make the proportion of P_i/H_i as nearly equal as possible to P_j/H_j, as $1 \le i < j \le 50$.

Huntington's Algorithm to compute Equal Proportions consist of 3 steps:

1. Assign one seat to each state.
2. Create an array of priority values, such that $P_1 \ge P_2 \ge P_3 \ge \ldots \ge P_{50}$

State 1	$\frac{P_1}{\sqrt{(1\cdot2)}}$	$\frac{P_1}{\sqrt{(2\cdot3)}}$	$\frac{P_1}{\sqrt{(3\cdot4)}}$	$\cdots$
State 2	$\frac{P_2}{\sqrt{(1\cdot2)}}$	$\frac{P_2}{\sqrt{(2\cdot3)}}$	$\frac{P_2}{\sqrt{(3\cdot4)}}$	$\cdots$
State *i*	$\frac{P_i}{\sqrt{(1\cdot2)}}$	$\frac{P_i}{\sqrt{(2\cdot3)}}$	$\frac{P_i}{\sqrt{(3\cdot4)}}$	$\cdots$
State 50	$\frac{P_{50}}{\sqrt{(1\cdot2)}}$	$\frac{P_{50}}{\sqrt{(2\cdot3)}}$	$\frac{P_{50}}{\sqrt{(3\cdot4)}}$	$\cdots$

3. Assign a seat to the state with the highest priority value in the table above. Repeat this step until all seats are assigned.

4. Proof Using Lagrange's Identity

Ideally, each representative will represents the same amount of people, that is, the equation $P_i/H_i = P_j/H_j$ holds for all states. Unfortunately, in real life, the chance of equality is negligible, therefore, it is needed to come up with as fair an allocation as possible. By rewriting the equation $P_i/H_i = P_j/H_j$ as

$$\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i} = 0,$$

we aim to minimize

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2$$

subject to the constraint, $\sum_{i=1}^{50} H_i = H$.

Now, by taking $a_i = \frac{P_i}{\sqrt{H_i}}$, $b_i = \sqrt{H_i}$, and $n = 50$, in Lagrange's Identity, we can see that

$$\left(\sum_{i=1}^{50}\frac{P_i^2}{H_i}\right)\left(\sum_{i=1}^{50} H_i\right) - \left(\sum_{i=1}^{50} P_i\right)^2 = \sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2.$$

By performing appropriate algebraic operations on the left-hand side of the equality, we can identify the following equation:

$$\begin{aligned}&\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2 \\ &= H\sum_{i=1}^{50}{P_i}^2 - P^2 - H\sum_{i=1}^{50}\left(\frac{(P_i)^2}{1\cdot2} + \frac{(P_i)^2}{2\cdot3} + \ldots + \frac{(P_i)^2}{(H_i-1)(H_i)}\right).\end{aligned}$$

To see the terms subtracted for each state i, we expand the last term on the right-hand side as follows:

$$\begin{aligned}&\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2 \\ &= \left(H\sum_{i=1}^{50}{P_i}^2 - P^2\right) - \frac{H(P_1)^2}{1\cdot2} - \frac{H(P_1)^2}{2\cdot3} - \ldots - \frac{H(P_1)^2}{(H_1-1)(H_1)} \\ &\quad - \frac{H(P_i)^2}{1\cdot2} - \frac{H(P_i)^2}{2\cdot3} - \ldots - \frac{H(P_i)^2}{(H_i-1)(H_i)} \\ &\quad - \frac{H(P_{50})^2}{1\cdot2} - \frac{H(P_{50})^2}{2\cdot3} - \ldots - \frac{H(P_{50})^2}{(H_{50}-1)(H_{50})}.\end{aligned}$$

Therefore, if we increase the number of seats assigned to state i from $H_i - 1$ to H_i, the amount by which the equation

$$\sum_{1\le i<j\le 50}\left(\frac{P_i}{\sqrt{H_i}}\sqrt{H_j} - \frac{P_j}{\sqrt{H_j}}\sqrt{H_i}\right)^2$$

decreases is given by $\frac{H{P_i}^2}{(H_i-1)(H_i)}$.

5. Conclusion

To see the similarity with the priority value table from step 2 of The Huntington's Algorithm, let's denote two points. Firstly, H is a common factor of all the terms being subtracted, hence it can be omitted. Secondly, all terms being subtracted are squares of positive real numbers, therefore, we can take a square root of all terms without changing the ordering.

The table below shows a real life example of a seat allocation following the 2020 Census. The first column shows the shares of seats based on the population of the state, while the second column shows the current number of seats for the state based on the method of equal proportions.

State	Share of seats	Number of seats
California	$\frac{39{,}538{,}223}{331{,}449{,}281}(435) \approx 51.891$	52
South Carolina	$\frac{5{,}118{,}425}{331{,}449{,}281}(435) \approx 6.718$	7
Rhode Island	$\frac{1{,}097{,}379}{331{,}449{,}281}(435) \approx 1.440$	2

References

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